

On the Solution to a Countable System of Equations Arising in Stochastic Processes

M.N. Katehakis, L.C. Smit, F.M. Spieksma

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Abstract

In this paper we develop a method to compute the solution to a countable (finite or infinite) set of equations that occurs in many different fields including Markov processes that model queueing systems, birth-and-death processes and inventory systems.

The method provides a fast and exact computation of the inverse of the matrix of the coefficients of the system. In contrast, alternative inverse techniques perform much slower and work only for finite size matrices.

Furthermore, we provide a procedure to construct the eigenvalues of the matrix under consideration.

1 Introduction

1.1 Motivation

In this paper we develop a method to compute the solution to a countable (finite or infinite) set of equations that occurs in many different fields and systems including Markov processes that model queueing systems, birth-and-death processes and inventory systems. For such systems in order to compute performance measures and other quantities of interest, it is often required to invert this matrix. A class of problems for which the inverse of this matrix must be computed is given in [11].

The method provides a fast and exact computation of the inverse of the matrix of the coefficients of the system. In contrast, alternative inverse techniques perform much slower and work only for finite size matrices. The more relevant alternative methods are discussed in this paper, for comparison purposes. It is shown that although some cover more general classes of matrices, the method developed in this paper provides a procedure for matrices of infinite size

and outperforms all alternative methods in speed. As far as we could find, there are no results in literature for countable sized matrices of this form. Existing algorithms for finite matrices perform in general slower than the method we provide.

Apart from the inverse, we have also identified a fast way to compute the eigenvalues of the matrix under consideration, starting with those of a much easier to analyze matrix, one with a birth-and-death structure. Knowing the values of these eigenvalues or just their bounds aids enormously in analysing the specifics of the models of interest.

This paper is organized as follows. First we will introduce some of the notation used in this paper and necessary to understand the review of other methods. Second in Section 1.3, we will identify some of the existing procedures and specify in what directions they overlap the method discussed in this paper. In Section 2 we formally provide the method with its specifications subdivided in four possible matrix forms. Specifically, see Algorithm 1, for the computation of the inverse. Next, in Section 3 we will exploit the structure of the matrix and derive some results regarding the eigenvalues of the specific matrix. Finally, in Section 4 we will give several applications wherein this matrix-structure appears naturally. Further we obtain Algorithm 2 and 3 for the cases of element homogenous B and birth-and-death processes with an absorbing state.

1.2 Preliminaries

In this paper we develop an efficient computation procedure for the solution vector x of size $\ell + 1$ ($\ell \leq \infty$) of a possibly countable system of linear equations of the form

$$xB = y, \quad (1)$$

where y is a vector of size $\ell + 1$, B is a matrix of size $(\ell + 1) \times (\ell + 1)$ and it has a structure of the form below:

$$B = \begin{bmatrix} -b_0^d - b_0^u & b_0^u & 0 & 0 & 0 & \cdots \\ b_1^d + b_1^z & -b_1^w & b_1^u & 0 & 0 & \cdots \\ b_2^z & b_2^d & -b_2^w & b_2^u & 0 & \ddots \\ b_3^z & 0 & b_3^d & -b_3^w & b_3^u & \ddots \\ b_4^z & 0 & 0 & b_4^d & -b_4^w & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad (2)$$

where in the above b_i^j ($i = 1, 2, 3, \dots$ and $j = d, w, u$) are non-negative real numbers and they

must satisfy the conditions below:

$$b_i^z + b_i^d + b_i^u = b_i^w, \quad (3)$$

$$b_i^w > 0, \quad (4)$$

$$b_0^d > 0. \quad (5)$$

To compute the solution of Eq. (1) one needs to compute the inverse of matrix B , i.e., the matrix $C = B^{-1}$. In Section 2 we provide constructive algorithms of computing C . These methods establish the existence of the inverse and aid to derive a way to truncate a matrix B with arbitrary precision when it is of infinite size.

In this paper we will distinguish 4 cases: we consider both a finite and an infinite sized matrix, and in both cases the transitions can be of a homogenous or non-homogenous structure. All of these four configurations follow a similar procedure, but have some technical differences that can enhance the speed and usability of the algorithm because of their specific structure.

1.3 Related Literature

The computation of the inverse of a general matrix is a procedure with a relatively high complexity when the size of the matrix is large. Therefore, there is an extensive literature devoted to studying this problem for matrices of special structure cf. [5], [4] and [13], [15].

In this section we discuss the main existing procedures to compute the inverse of matrix B of the form of Eq. (2), we will assume that ℓ is finite, since these procedures are mainly applicable for finite sized matrices only. Further, it is convenient to change notation and take B to be a matrix of size $\ell \times \ell$. Finally, we will decompose B as a combination of the following matrices:

$$B = \tilde{U} - W, \quad (6)$$

where

$$\tilde{U} = u\delta$$

with u a column vector and δ a row vector identically equal to zero and with a 1 as its first entry.

Computing the inverse of W can be done in various ways, since W has such a specific structure. Below we describe the main existing approaches in the literature that are based on using different properties of W .

First, we can analyse W from a tridiagonal form perspective (see [15]). A tridiagonal matrix is a band matrix (see [4] and [13]) with a bandwidth of 2; only the main diagonal and the two adjacent diagonals contain nonzero elements. In [15] a procedure to construct a LU decomposition is provided for tridiagonal matrices. However to multiply a lower diagonal matrix with an upper diagonal matrix still takes $\mathcal{O}(N^3)$ elementary operations. From [8],

page 61, we construct the inverse of B using the inverse of W , if it exists:

$$B^{-1} = W^{-1} + W^{-1}ua^{-1}\delta W^{-1}, \quad (7)$$

where the scalar a is defined as:

$$a = 1 - \delta W^{-1}u.$$

With a similar argument that has been used to prove that B is invertible, we can prove that the determinant of W is nonzero. The matrix multiplications that are used to construct B^{-1} using Eq. (7) from the separate elements are of $\mathcal{O}(\ell^2)$, as is shown below. We will compute B^{-1} in this lineup:

$$B^{-1} = W^{-1} + a^{-1}W^{-1}u\delta W^{-1} = W^{-1} + a^{-1}((W^{-1}u)\delta)W^{-1}$$

In this order, each of the multiplication steps requires at most $\mathcal{O}(\ell^2)$ operations. Therefore, if the inversion of W^{-1} takes at most $\mathcal{O}(\ell^2)$ operations, the inversion of B has the same order of operations, since the other action are vector times matrix multiplications.

Second, we note that if matrix B is element homogenous, the matrix W is (almost) of Toeplitz form (cf. [2, 3, 16]), only the lower right corner element is inconsistent with this classification, since the matrix is of finite dimension and the row sum of row ℓ is zero. We can make approximations to estimate the influence of this disruption. Many of the methods associated with Toeplitz matrices are devoted on finding the solution of the system $Wx = k$ where x is unknown and k a vector of size N . For the purposes that we have in mind, we will need to compute *every* element explicitly and most algorithms are not designed to do so. However, some of them are superfast (i.e. of $\mathcal{O}(N \log(N))$), and are useful in other related settings.

In [14], Section 3.3, a fast algorithm that can be applied to matrix W (an M-matrix in that paper) is provided for inverting a tridiagonal matrix in an efficient way. However, the procedure does not readily extend to countable matrices. Our method can be extended, since it uses the extra property that the row sum is zero for all rows except the first. Furthermore, the algorithms of [14] require that $b_i^d b_i^u \neq 0$ for all $i \in \{2, 3, \dots, N\}$, while the algorithms in this paper do not require these properties. The proposed algorithm of [14] requires to consider the all elements of the matrix, to express off diagonal elements in terms of a diagonal element. The diagonal elements are constructed by a recursive formula that uses the last element of this sequence to explicitly express the first elements. In the non-homogenous case, we encounter the same challenges for infinite state spaces as other algorithms. However, with our procedure we get a sequence that converges to zero, in contrast to the ones in [14]. This makes estimations slightly easier to manage. For algorithms regarding the homogenous structure we do not have this issue.

Also, the algorithms in the current paper readily extend to cases where the first two columns contain nonzero elements, possibly even non homogenous and not of the $c \cdot r$ form, something that the algorithm in [14] combined with Equation (7) can not do.

Besides their own algorithm, the article of Li et al. ([14]) contains a clarifying table (cf. Table 1, page 979 of that paper) with several different algorithms of other papers and their corresponding algorithms. It confirms that various other algorithms exist that have the same speed as the ones presented in this paper, but are not easily extendable to countable cases by their nature.

Fourth, matrix W satisfies the framework of a Hessenberg matrix, since it is the summation of a lower diagonal matrix and a tridiagonal matrix. In [9] an algorithm is provided to find two vectors x and y such that xy is the upper part of the inverse, i.e. $c_{ij} = x_i y_j$, and similarly for the lower part of the algorithm. However, this is not a computationally efficient algorithm, since it is designed to consider a much larger class of matrices.

To conclude this comparison, our method works for non homogenous cases, can be extended to matrices with two (or more) nonzero columns that are not in the same span, is correct for infinite state spaces and gives an explicit solution independent of the truncation size. Therefore we think that this algorithm is a major contribution to existing approaches.

2 Efficient Computation of the Inverse of Matrix B

In this section we will describe the four appearances of matrix B and describe their solution procedures. The general idea of the methods is similar, but their are specific differences and simplifications, specific for each case.

2.1 The Non Homogenous - Infinite Dimensional Case

In the sequel, for notational simplicity we let C denote the inverse of B , i.e., $C := B^{-1}$. The (i, j) th element of C will be denoted respectively by $c(i, j)$. We will use the notation B_i and B'_j (respectively C_i and C'_j) to denote the i^{th} row and j^{th} column of matrix B (respectively matrix C); $i, j = 0, 1, \dots, \ell$. Algorithm 1 below is based on the results of Propositions 1-5. It finds a matrix C that satisfies $CB = BC = I$, by computing recursively the so far unknown elements of the sequence $C'_0, C_0, C'_1, C_1, C'_2, C_2, \dots, C'_{n-1}, C_{n-1}, C'_n$ for increasing $n \geq 1$, as described below. The algorithm depends on the computation of a sequence of constants, γ_i , $i = 1, 2, \dots$, that can be computed if we assume that $b_i^d > 0$ for all $i = 0, 1, \dots$. In addition we first require that $b_i^u > 0$: for simplicity we present the algorithm under this assumption. In Proposition 6 we will show how γ_i , $i = 1, 2, \dots$, can be computed when $b_i^d = 0$ for some i . In addition, we will briefly explain how adjustments can be made to handle $b_i^u = 0$ for some i .

Algorithm 1.

At stage 0:

- a) The 0^{th} column of C (i.e., the column containing elements $c(i, 0)$) is computed using Eq. (8) of Proposition 1.

- b) All elements of the 0^{th} row of C (elements $c(0, j)$) are computed using Eqs. (9) and (10) of Proposition 2, where γ_1 is given by Eq. (15) of Proposition 3.

At stage $i = 1, 2, \dots, \ell$:

- a) For the i^{th} column C'_i , since the $c(0, i), \dots, c(i-1, i)$ have already been computed, we calculate the remaining by Eq. (16) of Proposition 4.
- b) For the i^{th} row C_i , since the $c(i, 0), \dots, c(i, i)$ have already been computed, we calculate the remaining by Eq. (17) of Proposition 5.

In the following propositions we will show that the algorithm above is correct.

Proposition 1. *The following is true for all $i \geq 0$:*

$$c(i, 0) = -1/b_0^d. \quad (8)$$

Proof. By considering the set of equations $BC'_0 = \delta'_0$ we find:

$$\begin{aligned} (-b_0^d - b_0^u)c(0, 0) + b_0^u c(1, 0) &= 1, \\ b_i^z c(0, 0) + b_i^d c(i-1, 0) - b_i^w c(i, 0) + b_i^u c(i+1, 0) &= 0, \text{ for } i \geq 1. \end{aligned}$$

Using the definition of b_i^w in Eq. (3) it is easy to see that for all $i \geq 0$, $c(i, 0) = -1/b_0^d$, is a solution that suffices and therefore has to be the unique solution. \square

Proposition 2 below shows that the elements $c(0, j)$ with $j > 0$ depend only on $c(0, 0)$. Note that $c(0, 0)$ is nonzero by its construction in Proposition 1.

Proposition 2. *Define*

$$\gamma_j := c(0, j)/c(0, 0). \quad (9)$$

There exist scalars ρ_j, η_j , with $j = 1, 2, \dots$ such that all constants γ_j can be recursively computed as a function of γ_1 as follows:

$$\gamma_j = \rho_j \gamma_1 + \eta_j, \quad (10)$$

where the constants ρ_j and η_j are given by, under the assumption that $b_j^d > 0$:

$$\rho_j = \frac{b_{j-1}^w \rho_{j-1} - b_{j-2}^u \rho_{j-2}}{b_j^d} \quad (11)$$

and

$$\eta_j = \frac{b_{j-1}^w \eta_{j-1} - b_{j-2}^u \eta_{j-2}}{b_j^d}, \quad (12)$$

with initial values: $\rho_1 = 1, \eta_1 = 0, \rho_2 = b_1^w/b_2^d, \eta_2 = -b_0^u/b_2^d$.

Proof. Equation (10) follows by the systems of equations $C_0 B = \delta$, without considering the first equality. It is easy to see that every equation has the form below, where $j > 0$:

$$b_{j-1}^u c(0, j-1) - b_j^w c(0, j) + b_{j+1}^d c(0, j+1) = 0. \quad (13)$$

The combination of this recursive structure and the ergodicity assumption (that ensures that $\lim_{j \rightarrow \infty} c(0, j) = 0$ for all j), allows us to express all elements $c(0, j)$ in terms of their preceding elements. Thus by substitution, we conclude that every element is a product of $c(0, 0)$ and a constant, depending on j that we denote as γ_j .

Eq. (10) uses Eq. (9) and follows from the observation that:

$$\gamma_2 = \frac{b_1^w \gamma_1 - b_0^u}{b_2^d},$$

and in general for $j \geq 3$:

$$\gamma_j = \frac{b_{j-1}^w \gamma_{j-1} - b_{j-2}^u \gamma_{j-2}}{b_j^d}. \quad (14)$$

It is clear that γ_2 is of the form described in the Proposition (since $\rho_2 = b_1^w/b_2^d$ and $\eta_2 = b_0^u/b_1^d$) and by substituting, γ_3 has this structure as well. An induction argument completes the proof. \square

Proposition 3 provides a method to calculate γ_1 , using the expressions derived above.

Proposition 3. *The scalar γ_1 can be calculated algebraically as follows:*

$$\gamma_1 = \frac{b_0^u - \sum_{j=1}^{\infty} b_j^z \eta_j}{b_1^d + \sum_{j=1}^{\infty} b_j^z \rho_j}. \quad (15)$$

Proof. To verify this expression for γ_1 , we consider the equation $C_0 B'_0 = 1$. The result follows immediately. \square

The next proposition provides a method of computing the under diagonal elements ($c(i, j)$, with $j = 1, 2, \dots$ and $i = j, j+1, \dots$) of C . We denote $\delta_{i,j}$ as a scalar that takes the value 1 if $i = j$, and 0 otherwise.

Proposition 4. *Assume that $b_i^u > 0$, for all $i = 0, 1, \dots$. The following is true for all elements $c(i, j)$ with $i \geq j \geq 1$.*

$$c(i, j) = \begin{cases} -\gamma_1(1/b_0^d + 1/b_0^u), & \text{for } i = j = 1, \\ \frac{1}{b_{i-1}^u}(-b_{i-1}^z c(0, j) - b_{i-1}^d c(i-2, j) + b_{i-1}^w c(i-1, j) + \delta_{i-1,j}), & \text{otherwise.} \end{cases} \quad (16)$$

Proof. To compute $c(1, 1)$ we use the equation $B_0 C'_1 = 0$, i.e.,

$$-(b_0^u + b_0^d)c(0, 1) + b_0^u c(1, 1) = 0,$$

and the statement is complete since by Proposition 2 we have $c(0, 1) = \gamma_1 c(0, 0) = -\gamma_1/b_0^d$.

To compute the remaining elements $c(i, j)$ of C , we use $B_{i-1} C'_j = \delta_{i-1, j}$. Indeed, the product of the $(i-1)^{th}$ row of B (where $i \geq 2$) and the j^{th} column of C (where $j \geq 1$) is the lefthand side of the equation below and the proof is complete.

$$b_{i-1}^z c(0, j) + b_{i-1}^d c(i-2, j) - b_{i-1}^w c(i-1, j) + b_{i-1}^u c(i, j) = \delta_{i-1, j}.$$

□

Proposition 5. *The following is true for $c(i, j)$ with $j > i \geq 1$ and $b_j^d > 0$.*

$$c(i, j) = \frac{b_{j-1}^w c(i, j-1) - b_{j-2}^u c(i, j-2) + \delta_{i+1, j}}{b_j^d}. \quad (17)$$

Proof. Equation (17) follows from the systems of equations $C_i B = \delta_i$, without considering the first $i-1$ equalities. Every equation has the form below, where $j > 0$:

$$b_{j-2}^u c(i, j-2) - b_{j-1}^w c(i, j-1) + b_j^d c(i, j) = \delta_{i+1, j}.$$

□

2.1.1 Zeros above and below the diagonal

In this section we discuss how Algorithm 1 can be extended to allow $b_i^d = 0$ and $b_i^u = 0$ to be zero. The elements $b_i^z = 0$ can always be zero and do not influence the procedure of the algorithm. When $b_i^d = 0$ for some i , Eq. (14) can not be used to compute the corresponding γ_i because of the devision by 0 that occurs. However, we next show that under the conditions in Eqs. (3)-(5) the matrix C still exists and this inverse can be computed by Algorithm 1 with a small modification to compute γ_i in the following way.

Let $I_0 = \{i_k, k = 1, \dots, \nu, \text{ such that } b_{i_k}^d = 0\}$ and let $i_0 = 1$ and $i_{\nu+1} = \infty$.

Proposition 6. *Assume that I_0 is not empty, then for $k = 0, 1, 2, \dots, \nu-1$:*

$$\gamma_{i_k} = \begin{cases} \frac{b_{i_{k+1}-1}^w \eta_{i_{k+1}-1} - b_{i_{k+1}-2}^u \eta_{i_{k+1}-2}}{-b_{i_{k+1}-1}^w \rho_{i_{k+1}-1} + b_{i_{k+1}-2}^u \rho_{i_{k+1}-2}}, & \text{if } i_{k+1} \neq i_k + 1, \\ \frac{b_{i_k-1}^u \gamma_{i_k-1}}{b_{i_k}^w}, & \text{if } i_{k+1} = i_k + 1. \end{cases} \quad (18)$$

For i_ν :

$$\gamma_{i_\nu} = \frac{b_0^u - \sum_{j=1}^{i_\nu-1} b_j^z \gamma_j - b_1^d \gamma_1 - \sum_{j=i_\nu}^{\infty} b_j^z \eta_j}{\sum_{j=i_\nu}^{\infty} b_j^z \rho_j}. \quad (19)$$

The remaining components of γ are constructed for $j \in \{i_k + 1, \dots, i_{k+1} - 1\}$ as:

$$\gamma_j = \rho_j \gamma_{i_k} + \eta_j.$$

where the vectors ρ and η are as before in Eq. (11) and Eq. (12), specified below, with $j = 1, 2, \dots$

$$\rho_j = \begin{cases} 1, & \text{if } b_j^d = 0 \text{ or if } j = 1, \\ \frac{b_{j-1}^w}{b_j^d}, & \text{if } b_j^d \neq 0 \text{ and } b_{j-1}^d = 0, \text{ or if } j = 2, \\ \frac{b_{j-1}^w \rho_{j-1} - b_{j-2}^u \rho_{j-2}}{b_j^d}, & \text{if } b_j^d \neq 0 \text{ and } b_{j-1}^d \neq 0, j \geq 3, \end{cases}$$

and:

$$\eta_j = \begin{cases} 0, & \text{if } b_j^d = 0 \text{ or if } j = 1, \\ \frac{-b_{j-2}^u \gamma_{j-2}}{b_j^d}, & \text{if } b_j^d \neq 0 \text{ and } b_{j-1}^d = 0, \text{ or if } j = 2, \\ \frac{b_{j-1}^w \eta_{j-1} - b_{j-2}^u \eta_{j-2}}{b_j^d}, & \text{if } b_j^d \neq 0 \text{ and } b_{j-1}^d \neq 0, j \geq 3, \end{cases}$$

where $\gamma_0 = 1$.

Proof. We show how to compute $\gamma_1, \gamma_2, \dots, \gamma_{i_1}$. Consider the steady state equations given in Eq. (13): only the equation corresponding to $i = i_1 - 1$ changes to

$$b_{i_1-2}^u \gamma_{i_1-2} = b_{i_1-1}^w \gamma_{i_1-1}.$$

Since there is no change in the formula for $i < i_1 - 1$, Eqs. (10) hold for $i < i_1 - 1$, with ρ and η defined as above, from which we obtain the equations below (for $i = i_1 - 1$ and $i = i_1 - 2$ respectively):

$$\gamma_{i_1-1} = \rho_{i_1-1} \gamma_1 + \eta_{i_1-1}$$

and similarly for γ_{i_1-2} :

$$\gamma_{i_1-2} = \rho_{i_1-2} \gamma_1 + \eta_{i_1-2}.$$

Combining the three equations above, we find the expression of Eq. (18) for γ_1 .

Eq. (19) follows as before from $C_0 B'_0 = 1$, using the computed γ_i by Eq. (18), $i = 1, \dots, i_\nu - 1$.

To find the subsequent values of γ_i we repeat this process starting at the beginning of the next segment of the set $\{\gamma_{i_1+1}, \dots, \gamma_{i_2}\}$. The proof is complete noting that the case $\nu = \infty$ corresponds to an infinite set $\{\gamma_{i_\nu-1}, \dots\}$ on which the approach of the algorithm is applied. \square

In the sequel when we refer to Algorithm 1 we include the possible modification applied per Proposition 6.

What adjustments need to be made to compute the remaining elements when a division by zero is required for the algorithm (i.e. $b_{i-1}^u = 0$ in Eq. (16)), are very similar to the procedure described in Proposition 6. We ‘shift’ away from this equation and use the next equations, to express following elements in terms of the one that was not computable because of the undefined division. This procedure happens horizontally for upper diagonal elements when $b_i^d = 0$, and vertically for under diagonal elements when $b_j^u = 0$.

Note that when $b_i^u = 0$ then $c(k, l) = 0$ for $1 \leq k \leq i$ and $l \geq i + 1$.

2.1.2 Multiple nonzero columns

One way to enlarge the class of matrices for which the method above can be applied is by allowing other columns to be nonzero for every element. Matrix B will now have the following form:

$$B = \begin{bmatrix} -b_0^d - b_0^u & b_0^u + b_0^{z2} & 0 & 0 & 0 & \dots \\ b_1^d + b_1^{z1} & -b_1^w + b_1^{z2} & b_1^u & 0 & 0 & \dots \\ & b_2^{z1} & b_2^d + b_2^{z2} & -b_2^w & b_2^u & 0 & \ddots \\ & b_3^{z1} & b_3^{z2} & b_3^d & -b_3^w & b_3^u & \ddots \\ & b_4^{z1} & b_4^{z2} & 0 & b_4^d & -b_4^w & \ddots \\ & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \quad (20)$$

As before, the row sum is zero for every row, i.e. for all i :

$$b_i^{z1} + b_i^{z2} + b_i^d + b_i^u = b_i^w.$$

Algorithm 1 can still be applied, the only adjustments that need to be made are briefly described below. We will leave the details to the reader. The main difference with the original algorithm is that in this case, we express all elements of the top row in terms of the top row element corresponding to the nonzero column with the highest index.

As long as the number of nonzero remains small, the order of the algorithm remains the same, only the normalization takes some extra steps. When the number of nonzero columns becomes countable, the algorithm loses its computational advantage and other methods might be preferable.

Another approach is to use the Toeplitz equation, cf. Eq. (7), when the second column is a multiple of the first column. This is a necessary condition to use this equation.

2.2 The Non Homogenous - Finite Dimension Case

The case in which $\ell + 1 < \infty$, may happen due to modeling or truncation. No adjustments are needed in the formulation of Algorithm 1, since the only difference is the altered condition at the boundary:

$$b_\ell^w = b_\ell^d + b_\ell^z,$$

i.e., the row sum of the final row is zero. The formulas of Propositions 2 - 6 are adjusted so that the corresponding summations are up to ℓ where ℓ is finite.

2.3 The Homogenous - Infinite Dimension Case

In this paragraph we define the following subclass of matrices B that possess the structure described in Eq. (2) and the additional structure of the definition below.

Definition 1. *B is called element homogeneous when the following relations hold*

$$b_i^d = b^d, b_i^u = b^u, b_i^z = b^z \text{ for all } i.$$

We note that when B is *element homogeneous* then necessarily the following also holds: $b_i^w = b^w$, for all i .

In the rest of this section we denote the constant γ_1 as γ and derive the following result. We assume that none of the elements given are zero to avoid trivialities and more tedious notation. First we derive the following result regarding the scalar γ .

Proposition 7. *When B has a homogenous structure and the size ℓ is countable, the following is true for all $j \geq 1$:*

$$\gamma_j = \gamma^j, \tag{21}$$

where

$$\gamma = \frac{b^w - \sqrt{(b^w)^2 - 4b^u b^d}}{2b^d}. \tag{22}$$

Proof. The proof follows by considering the systems of equations $C_i B_j' = 0$ for $j = i + 1, i + 2, \dots$. All equalities have the form below.

$$b^u c(i, j - 1) - b^w c(i, j) + b^d c(i, j + 1) = 0.$$

Because of the homogenous structure, it is clear that the corresponding elements of the row C_0 have a product form with a constant factor γ , e.g. for $j = i$:

$$b^u c(i, j) - b^w \gamma c(i, j) + b^d \gamma^2 c(i, j) = 0,$$

thus in other words:

$$c(i, j) = \gamma^j c(i, i). \tag{23}$$

When, solving for γ the above simplifies to $b^d\gamma^2 - b^w\gamma + b^u = 0$. This quadratic equation has a unique solution between 0 and 1 given by Eq. (22). This solution is greater than 0 since $\sqrt{(b^w)^2 - 4b^ub^d} < \sqrt{(b^w)^2} = b^w$. This γ is smaller than 1 since:

$$\begin{aligned}\gamma &= \frac{b^w - \sqrt{(b^w)^2 - 4b^ub^d}}{2b^d} < \frac{b^w \sqrt{(b^w)^2 - 4b^ub^d - 4b^zb^d}}{2b^d} \\ &= \frac{b^w - \sqrt{(b^w)^2 + 4(b^d)^2 - 4b^wb^d}}{2b^d} \\ &= \frac{b^w - \sqrt{(b^w - 2b^d)^2}}{2b^d} \\ &= \frac{2b^d}{2b^d} = 1.\end{aligned}$$

□

When B is element homogenous, we do not need Proposition 3 to compute C , since $\gamma_1 = \gamma$ and computable by Proposition 7. Proposition 4 remains the same when B is element homogenous. In addition, the expression for the computation of the under diagonal elements can also be simplified. Using both expressions, we can even directly express the diagonal elements in terms of its predecessor.

The above leads to the following considerable simplification of Algorithm 1.

Algorithm 2. *Computation of $C := B^{-1}$ for an element homogeneous, countable, matrix B .*

At stage 1:

- a) Calculate γ using Eq. (22).
- b) Calculate ψ using Eq. (26).
- c) The remaining elements of the 0^{th} row of C are computed using Proposition 7 and in particular Eq. (23) in that proposition.

At stage $i = 1, 2, \dots, \ell$:

- a) The diagonal element $c(i, i)$ is computed by Eq. (27).
- b) The elements of C_i , the i^{th} row of C to the right of the diagonal are computed using Proposition 7 and in particular Eq. (23) in that proposition.
- c) The under diagonal elements of C'_i , the i^{th} column of C are computed using Eq. (25) of Proposition 8.

To summarize, the elements $c(i, j)$ of C can be calculated as follows:

$$c(i, j) = \begin{cases} \frac{-1}{b^d} & \text{if } j = i = 0, \\ \frac{-1 + b_u((1 - \psi)c(0, i - 1) + \psi c(i - 1, i - 1))}{b_w - b_d \gamma} & \text{if } j = i \geq 1, \\ \gamma^{j-i} c(i, i) & \text{if } j \geq i + 1, \\ (c(j, j) - c(0, j))\psi^{i-j} + c(0, j) & \text{if } j \leq i - 1. \end{cases} \quad (24)$$

For a feasible way to compute all elements, we refer to the order in Algorithm 2.

Below we will prove that Algorithm 2 is correct. As part of this algorithm, the under diagonal elements of column C'_i are expressed in terms of its first element $c(i, 0)$ and its diagonal element $c(i, i)$. This calculation is given in the proposition below.

Proposition 8. *The under diagonal elements of C can be calculated as follows:*

$$c(i, j) = (c(j, j) - c(0, j))\psi^{i-j} + c(0, j), \quad (25)$$

where

$$\psi = \frac{b^w - \sqrt{(b^w)^2 - 4b^u b^d}}{2b^u} = \gamma b^d / b^u, \quad (26)$$

and where ψ is between 0 and 1.

Proof. We consider the equation $B'_i C_j = 0$, with $i \geq j + 1$:

$$b^z c(0, j) + b^d c(i - 1, j) - b^w c(i, j) + b^u c(i + 1, j) = 0.$$

Secondly, we define $d(i, j) := c(i, j) - c(0, j)$ for all $i \geq j$. Then it is easy to see that the above is the same as:

$$b^d d(i - 1, j) - b^w d(i, j) + b^u d(i + 1, j) = 0.$$

Like in the proof of Proposition 7 these elements $d(i, j)$ have a product form in i with a constant factor ψ , for all $i \geq j + 1$:

$$b^d d(i - 1, j) - b^w \psi d(i, j) + b^u \psi^2 d(i + 1, j) = 0,$$

thus in other words:

$$d(i, j) = \psi^{i-j} d(j, j).$$

When solving for ψ , this quadratic equation has a unique solution between 0 and 1, given by Eq. (26). Next, substituting $d(i, j) = c(i, j) - c(0, j)$ we get:

$$d(i, j) = c(i, j) - c(0, j) = \psi^{i-j} (c(j, j) - c(0, j)).$$

and the proof is complete. \square

Since ψ and λ are expressed explicitly, we can calculate the diagonal elements using Equation Eq. (27) below. This is true by isolating $c(i, i)$ in $C_i B'_i = 1$.

$$c(i, i) = (-1 + b_u((1 - \psi)c(0, i - 1) + \psi c(i - 1, i - 1)))/(b_w - b_d \gamma) \quad (27)$$

In the next proposition we derive an interesting result regarding the convergence of the diagonal elements, where we abbreviate $D := b_w^2 - 4b_u b_d$. Note that $D > 0$.

Proposition 9. *The diagonal elements of C converge to:*

$$\lim_{i \rightarrow \infty} c(i, i) = \frac{-1}{\sqrt{D}}.$$

Proof. First, it is important to note that by definition, $b^d \gamma^2 - b^w \gamma + b^u = 0$, and thus:

$$b^d \gamma^2 = b^w \gamma - b^u.$$

We want to show that the diagonal elements $c(i, i)$ converge to a (negative) constant, for all $i = 1, 2, \dots$. Therefore we define the difference $v(i)$ between two subsequent diagonal elements as:

$$v(i) := c(i, i) - c(i - 1, i - 1).$$

In addition, we use an alternative expression for $c(i, i)$ in the equality $B_{i-1} C'_i = 0$, given in Eq. (28) below.

$$c(i, i) = (-b_z c(0, i) - b_d \gamma^2 c(i - 2, i - 2) + b_w \gamma c(i - 1, i - 1))/b_u. \quad (28)$$

We find a recursive expression for $v(i)$, using Eq. (28) and the following steps:

$$\begin{aligned} -b_z c(0, i) &= \gamma^i b_z / b_d = b_u c(i, i) - b_w \gamma c(i - 1, i - 1) + b_d \gamma^2 c(i - 2, i - 2) \\ &= b_u (c(i, i) - c(i - 2, i - 2)) - b_w \gamma (c(i - 1, i - 1) - c(i - 2, i - 2)) \\ &= b_u (v(i) - v(i - 1)) - b_w \gamma v(i - 1), \end{aligned}$$

and thus:

$$v(i) = \gamma^i b_z / b_d b_u + \frac{b_w \gamma - b_u}{b_u} v(i - 1). \quad (29)$$

In the sequel we will use that $\gamma^i \rightarrow \infty$ and that:

$$\begin{aligned} \frac{b_w \gamma - b_u}{b_u} &= b_d \gamma^2 / b_u, \\ &= \psi \gamma. \end{aligned}$$

Note that since both ψ and γ are between 0 and 1, their product is in the same interval. We can use all of the above to prove the convergence of $c(i, i)$ by considering the absolute value of the difference function $v(i)$.

$$\begin{aligned}
\lim_{i \rightarrow \infty} |v(i)| &= \lim_{i \rightarrow \infty} |\gamma^i b_z / (b_d b_u) + \frac{b_w \gamma - b_u}{b_u} v(i-1)| \\
&= \frac{b_d \gamma^2}{b_u} \lim_{i \rightarrow \infty} |\gamma^{i-2} b_z / b_d^2 + v(i-1)| \\
&\leq \psi \gamma \lim_{i \rightarrow \infty} |\gamma^{i-2} b_z / b_d^2| + \lim_{i \rightarrow \infty} |v(i-1)| \\
&= \psi \gamma \lim_{i \rightarrow \infty} |v(i-1)| \\
&= \psi \gamma \lim_{i \rightarrow \infty} |v(i)|,
\end{aligned}$$

and thus zero. Since we have found that the series converges, we can explicitly express this limit and complete the proof.

$$\begin{aligned}
\lim_{i \rightarrow \infty} c(i, i) &= \lim_{i \rightarrow \infty} (-1 + (b_u + b_z - b_u \psi) \gamma^i c(0, 0) + b_d \gamma c(i-1, i-1)) / (b_w - b_u \psi) \\
&= -1 / (b_w - b_u \psi) + b_d \gamma / (b_w - b_u \psi) \lim_{i \rightarrow \infty} c(i, i) \\
&= -1 / (b_w - b_d \gamma - b_u \psi) \\
&= \frac{-1}{\sqrt{D}}.
\end{aligned}$$

□

2.4 The Homogenous - Finite Dimension Case

In this section we consider the element homogenous and finite version of matrix B . To be able to use Algorithm 2 and thus to compute the row and column independent scalars γ and ψ , it is necessary that the state space is truncated in a specific way. This truncation is described in the proposition below.

Proposition 10. *Consider a finite and element homogenous matrix B of size $\ell \times \ell$ satisfying Eq. (2). If:*

$$B(\ell, \ell) = -b^u / \gamma$$

and

$$B(1, \ell) = b^u / \gamma - b^d$$

then $c(i, j)$ can be computed per Eq. (24) where γ is defined as in Eq. (22) and ψ as in Eq. (26).

Proof. When considering $C_i B'_\ell = 0$ with $0 \leq i \leq \ell - 1$ we find that:

$$b^u c(i, \ell) - b^u / \gamma c(i, \ell) = 0$$

and thus that $c(i, \ell) = \gamma c(i, \ell - 1)$. All elements $c(i, j)$ can now be expressed in terms of γ and its predecessor analogous to Proposition 7.

When considering $B_\ell C'_j = 0$ with $0 \leq i \leq \ell - 1$ and the observation that $b^u/\gamma = b^d/\psi$ we find that:

$$(b^d/\psi - b^d)c(0, j) + b^d c(\ell - 1, j) - b^d/\psi c(\ell, j) = 0.$$

Rewriting gives that:

$$b^d(c(\ell, j) - c(0, j)) = \psi b^d(c(\ell - 1, j) - c(0, j)).$$

All elements can now be expressed in terms of ψ , its predecessor and $c(0, j)$, analogous to Proposition 8. \square

When an alternative truncation of B is chosen or provided, the elements γ and ψ turn out to be row and column dependent and therefore we refer to the solution procedure of a non element homogenous finite matrix B .

It is clear that to computation of each element only requires a linear number of steps. Therefore the complexity of Algorithms 1 and 2 is $\mathcal{O}(\ell^2)$.

3 The eigenvalues of Matrix B

In this section we will state properties for the eigenvalues $\{\lambda_i^b\}_{i=1, \dots, \ell+1}$ of matrix B , valid for the two finite appearances of the matrix described in the previous sections.

We will use the structure of matrix W , introduced in Eq. (6). In general we do not require that $w_{i,i-1}w_{i-1,i} > 0$. When this property is violated, then most of the analysis below does not hold. For this section we therefore do assume this inequality to hold. We will make use of the following well-known lemma, proved in several papers.

Proposition 11. *Consider a finite tridiagonal, diagonal matrix W with entries w_{ij} where the product $w_{i,i+1}w_{i-1,i} > 0$ and $w_{ii} < 0$ for all i and $w_{i,i-1} + w_{i,i} + w_{i,i+1} = 0$. Its eigenvalues $\{\lambda_i\}_{i=1, \dots, \ell+1}$ are negative, real and distinct.*

Proof. One of the more complicated proofs of this statements uses Sturm sequences. (cf. [7]) Other proofs involve diagonalization, or follow the line of Meurant (cf. [17]). \square

Using the results in the paper of Alexanderian [1], Remark 3.4 and Remark 3.6 we know that for a small perturbation in matrix B the eigenvalues lie in a balls of radius ϵ around the eigenvalues of W . The corresponding set of eigenvectors $\{c_1, c_2, \dots, c_{\ell+1}\}$ of W is therefore a complete set of eigenvectors for A and therefore spans the space $\mathbb{R}^{\ell+1}$. The first element of c_i can not be zero: the equations of $Ac = \lambda c$ imply iteratively that all entries of c_i are zero and thus that c_i is a null-vector.

Below we prove a result regarding the eigenvalues of B .

Theorem 1. *When matrix B is finite, all its eigenvalues have a negative real part.*

Proof. Since B is invertible (cf. [12]) all eigenvalues are nonzero. By the Gershgorin circle theorem we know that every eigenvalue of B lies within at least one of the Gershgorin discs $D(b_{ii}, R_i)$ for all $i \in \{1, \dots, \ell+1\}$ where $R_i = \sum_{j \neq i} b_{ij}$ and by definition the diagonal elements $b_{ii} < 0$. Since $R_i < -b_{ii}$ every $D(b_{ii}, R_i)$ is non positive. \square

One possible way to construct matrix B from matrix W is by adding a linear combination of the eigenvectors (since W has full rank, see Proposition 11) to the first column:

$$B = W + (n_1 c_1 + n_2 c_2 + \dots + n_{\ell+1} c_{\ell+1}) \delta,$$

where δ is a row vector with $\delta(0) = 1$ and $\delta(i) = 0$ elsewhere; the numbers n_i are scalars, possibly zero. By reordering and renaming such that $\{n_1, n_2, \dots, n_M\} \neq 0$ and by norming the eigenvectors c_i at size $1/n_i$, we get without loss of generality that:

$$B = W + (c_1 + c_2 + \dots + c_M) \delta, \quad (30)$$

with c_i a normed (renamed) eigenvector. We prove the following proposition, regarding matrix $A_1 := A + \alpha_1 c_1 \delta$, for any matrix A (not necessarily a tridiagonal matrix), where we use the same notation for the eigenvalues and eigenvectors of A as those of W and scalar α_1 is chosen such that $\alpha_1 \neq \frac{\lambda_i - \lambda_1}{c_1(0)}$ for all i .

Proposition 12. *Matrix A_1 has eigenvalues $\{\lambda_1 + \alpha_1 c_1(0), \lambda_2, \dots, \lambda_{\ell+1}\}$ and eigenvectors $\{c_1, c_2 + x_1 c_1, \dots, c_{\ell+1} + x_{\ell+1} c_1\}$, where:*

$$x_i = \frac{\alpha_1 c_i(0)}{\lambda_i - \lambda_1 - \alpha_1 c_1(0)}.$$

Proof. In the derivation below we use this equality:

$$c_1 = (A - \lambda I)^{-1} (A - \lambda I) c_1 = (A - \lambda I)^{-1} (A c_1 - \lambda c_1) = (A - \lambda I)^{-1} (\lambda_1 - \lambda) c_1.$$

The characteristic polynomial for A_1 is as follows:

$$\begin{aligned} |A_1 - \lambda I| &= |A + \alpha_1 c_1 \delta - \lambda I| = |A - \lambda I + \alpha_1 c_1 \delta| \\ &= |A - \lambda I| (1 + \alpha_1 \delta (A - \lambda I)^{-1} c_1) \\ &= \left(\pm \prod_{i=1}^{\ell+1} (\lambda_i - \lambda) \right) \left(1 + \frac{\alpha_1 \delta c_1}{\lambda_1 - \lambda} \right) \\ &= \left(\pm \prod_{i=2}^{\ell+1} (\lambda_i - \lambda) \right) (\lambda_1 + \alpha_1 c_1(0) - \lambda). \end{aligned}$$

The roots of this polynomial are the eigenvalues of A_1 and equal to $\{\lambda_1 + \alpha_1 c_1(0), \lambda_2, \dots, \lambda_{\ell+1}\}$, denoted as $\{\lambda_1^1, \lambda_2^1, \dots, \lambda_{\ell+1}^1\}$.

We denote the corresponding eigenvectors of A_1 by $\{c_1^1, \dots, c_{\ell+1}^1\}$ and they are computed as below. By definition, $(A_1 - \lambda_1^1 I)c_1^1 = 0$ and we derive:

$$\begin{aligned} (A_1 - \lambda_1^1 I)c_1^1 &= (A + \alpha_1 c_1 \delta - (\lambda_1 + \alpha_1 c_1)I) c_1^1 \\ &= (A - \lambda_1 I)c_1^1 + \alpha_1 c_1 \delta c_1^1 - \alpha_1 c_1^1 \delta c_1 \\ &= 0. \end{aligned}$$

The solution of this system of equations is $c_1^1 = c_1$. This is the unique solution, since it solves a set of $\ell + 1$ independent linear equations with $\ell + 1$ variables.

Next, we show that for all $i \geq 2$, c_i^1 is a linear combination of c_i and c_1 , i.e. $c_i^1 = x_i c_1 + c_i$. We consider $(A_1 - \lambda_i^1 I)c_i^1 = 0$.

$$\begin{aligned} (A_1 - \lambda_i^1 I)c_i^1 &= (A + \alpha_1 c_1 \delta - \lambda_i I) c_i^1 \\ &= (A - \lambda_i I)(x_i c_1 + c_i) + \alpha_1 c_1 (x_i c_1(0) + c_i(0)) \\ &= x_i A c_1 - x_i \lambda_i c_1 + \alpha_1 x_i c_1(0) c_1 + \alpha_1 c_i(0) c_1 \\ &= x_i (\lambda_1 - \lambda_i + \alpha_1 c_1(0)) c_1 + \alpha_1 c_i(0) c_1 \\ &= 0. \end{aligned}$$

Thus choosing x_i as below solves this equation:

$$x_i = \frac{-\alpha_1 c_i(0)}{\lambda_1 - \lambda_i + \alpha_1 c_1(0)} = \frac{\alpha_1 c_i(0)}{\lambda_i^1 - \lambda_1^1}.$$

□

Corollary 1. *If the eigenvalues of an arbitrary matrix A are real and α_1 is real, then the eigenvalues of A_1 are real.*

As is described above, B is a combination of matrix W and a finite number M of well normed eigenvectors added to its first column. We want to use the result of Proposition 12 to construct the eigenvalues of B from those of W . However, after the addition of an eigenvector, the other eigenvectors change. Adding another eigenvector c_i of W does not satisfy the conditions to use Proposition 12 anymore, since c_1 is not an eigenvalue of the constructed matrix. The eigenvalues will change differently from the way described in that proposition.

To use Proposition 12 iteratively, we define a sequence of matrices $W_i, i = 1, 2, \dots, M$, as follows:

$$W_i := W_{i-1} + \alpha_i c_i^{i-1}, \quad (31)$$

where $W_0 = W$. We will denote the eigenvalues and eigenvectors of W_i as $\{\lambda_j^i\}_{j=1,\dots,\ell+1}$ and $\{c_j^i\}_{j=1,\dots,\ell+1}$. An intuitive but wrong way to construct B would be to choose $\alpha_i = 1$ for all i , as described above. The eigenvectors of W_i differ from those of W_{i-1} , and are in fact linear combinations of those of W_i . Thus we need to choose the factors α_i in such a way that for all i , the vector c_i is completely added to W . To ensure this, and thus that $B = W_M$ we add a fraction of vector c_i^i in such a way that the ‘remaining’ portion of that vector is spread out over the eigenvectors of W_i .

In other words, we are looking for scalars α_i such that the following set of equalities holds:

$$B = W + \sum_{j=1}^M c_j = \dots = W_i + \sum_{j=i+1}^M c_j^i = \dots = W_M.$$

Displaying W_i in terms of W by using Eq. (31) repeatedly, the dependence on α_i shows:

$$B = W + \sum_{j=1}^M c_j = \dots = W + \sum_{j=1}^i \alpha_j c_j^{j-1} + \sum_{j=i+1}^M c_j^i = \dots = W + \sum_{j=1}^M \alpha_j c_j^{j-1}.$$

This simplifies to the following for all $i = 1, 2, \dots, M$:

$$\sum_{j=1}^{i-1} \alpha_j c_j^{j-1} + \sum_{j=i}^M c_j^{i-1} = \sum_{j=1}^i \alpha_j c_j^{j-1} + \sum_{j=i+1}^M c_j^i$$

and thus:

$$(1 - \alpha_i) c_i^{i-1} = \sum_{j=i+1}^M (c_j^i - c_j^{i-1}). \quad (32)$$

From Proposition 12 and from the assumption we learned that $\lambda_j^i = \lambda_j^{i-1} = \dots = \lambda_j$ for all $j \geq i$. The eigenvectors c_j^i of W_i are a linear combination of c_j^{i-1} and c_i^{i-1} as follows:

$$c_j^i = c_j^{i-1} + \frac{\alpha_i c_j^{i-1}(0)}{\lambda_j - \lambda_i - \alpha_i c_i^{i-1}(0)} c_i^{i-1}. \quad (33)$$

Combining Eq. (32) and (33), we express α_i as a solution of the vector equations:

$$(1 - \alpha_i) c_i^{i-1} = \sum_{j=i+1}^M \left(\frac{\alpha_i c_j^{i-1}(0)}{\lambda_j - \lambda_i - \alpha_i c_i^{i-1}(0)} \right) c_i^{i-1}.$$

This set of dependent relations reduces to a single equation:

$$1 - \alpha_i = \alpha_i \sum_{j=i+1}^M \frac{c_j^{i-1}(0)}{\lambda_j - \lambda_i - \alpha_i c_i^{i-1}(0)}. \quad (34)$$

This equation has at least one (possible non-real) solution, since $\lambda_j - \lambda_i \neq 0$. We derive the following corollary regarding the eigenvalues of matrix B .

Corollary 2. *The eigenvalues λ_i^b of B are:*

$$\lambda_i^b = \lambda_i + \alpha_i c_i^{i-1},$$

where λ_i is an eigenvalue of W , c_i^{i-1} is constructed recursively via Equation (33) and α_i is the solution of Equation (34).

Note that the eigenvalues of B are not necessarily distinct, despite the fact that those of W are. Although we have assumed that we start with a tridiagonal matrix W this is not necessary for the derivation above. Therefore, as long as the perturbation in the first column can be expressed as a linear combination of eigenvectors the procedure above works to find the eigenvalues of matrix B .

We derive from Corollary 1 and 2 that if all α_i are real, then the eigenvalues of B are real. Below we will provide a sufficient condition for this to be true. Therefore, without loss of generality we will reorder and rename the eigenvectors and eigenvalues such that $\lambda_1 < \lambda_2 < \dots < \lambda_M < 0$.

We emphasize that the scalars α_i only depend on the first entries of eigenvectors c_i , cf. Eq. (34). For notational convenience in the sequel, we rewrite Eq. (34) as a function of y :

$$f_i(y) = 1 - y - y \sum_{j=i+1}^M \frac{c_j^{i-1}(0)}{\lambda_j - \lambda_i - y c_i^{i-1}(0)}. \quad (35)$$

We are interested in whether the function $f_i(\cdot)$ has a real root or not. If it has a real root, then we can use this root as a solution α_i to construct the eigenvalues of W_i . By Proposition 12 all these eigenvalues are real, if those of W_{i-1} are real. In the proposition below we identify a sufficient condition for $f_i(\cdot)$ to have a real root. We define $S_{i,j} := (\lambda_j - \lambda_i)/c_i^{i-1}(0)$, the singularities of $f_i(\cdot)$. Since $\lambda_{j+1} > \lambda_j$ for all j this sequence is strictly increasing and positive (decreasing and negative) in j when $c_i^{i-1}(0) > 0 (< 0)$. As is reasoned before, $c_i^{i-1}(0) \neq 0$.

Proposition 13. *Suppose that $c_i^{i-1}(0)$ and $c_{i+1}^{i-1}(0)$ are both positive. Then $f_i(\cdot)$ has at least one real positive root.*

Proof. By the assumption in the proposition, the quotient $\frac{y c_{i+1}^{i-1}(0)}{\lambda_{i+1} - \lambda_i - y c_i^{i-1}(0)} > 0$ when $0 < y < S_{i,i+1}$, since $\lambda_{i+1} - \lambda_i > y c_i^{i-1}(0)$ and $c_{i+1}^{i-1}(0) > 0$. This quotient is the dominant term near the singularity $S_{i,i+1}$ and thus $\lim_{y \uparrow S_{i,i+1}} f_i(y) = -\infty$. Since $f_i(0) = 1$, there is at least one positive real root on the interval $(0, S_{i,i+1})$. \square

Proposition 14. *When $c_i^{i-1}(0) < 0$ there is at least one real positive root of $f_i(\cdot)$.*

Proof. Recall that $S_{i,i+1}$ is the largest singularity and smaller than zero when $c_i^{i-1}(0) < 0$. Therefore, $f_i(\cdot)$ is continuous on $[0, \infty)$. Since $f_i(0) = 1$ and $\lim_{y \rightarrow \infty} f_i(y) = -\infty$ the function $f_i(\cdot)$ has at least one positive real root. \square

In the proposition below we will provide a condition that ensures that all scalars α_i are real.

Proposition 15. *All scalars α_i are real when $c_j(0) < 0$ for all $j \leq I$ and $c_j(0) < 0$ for all $j > I$ with $I \in \{0, 1, 2, \dots, M\}$.*

Proof. With this sequence of first elements, either $c_1(0) < 0$ or $c_1(0), c_2(0) > 0$. We know by Proposition 13 and Proposition 14 that there is at least one real root for $f_0(\cdot)$, and moreover that it is positive. Choose α_1 to be the smallest positive real root of $f_i(\cdot)$.

Eq. (33) implies the following recursion for the first element of c_j^1 :

$$c_j^1(0) = c_j(0) \left(1 + \frac{\alpha_1 c_1(0)}{\lambda_j - \lambda_1 - \alpha_1 c_1(0)} \right) = c_j(0) \left(\frac{\lambda_j - \lambda_1}{\lambda_j - \lambda_1 - \alpha_1 c_1(0)} \right).$$

If $c_1(0) < 0$, then $\frac{\lambda_j - \lambda_1}{\lambda_j - \lambda_1 - \alpha_1 c_1(0)} > 1$, since $\alpha_1 > 0$.

If $c_1(0) > 0$, then $\frac{\lambda_j - \lambda_1}{\lambda_j - \lambda_1 - \alpha_1 c_1(0)} > 0$, since $\alpha_1 \in (0, S_{1,2})$.

Thus $c_j^1(0)$ has the same sign as $c_j(0)$ for all $j = 2, 3, \dots, M$. The provided structure of this elements ensures that $f_1(\cdot)$ has at least one positive real root α_1 , since now $c_2^1(0) < 0$ or $c_2^1(0), c_3^1(0) > 0$. We can repeat this argument and construct all positive real roots α_i for all i . \square

We conjecture that the sequence $c_i(0)$ satisfies the condition, given in Proposition 15 for the case when B is sub-stochastic matrix where the row sum is negative only in the first row. Numerical examples confirm this conjecture, but we leave its proof as an open problem.

4 Applications

Matrix B discussed in the previous sections occurs in many models, particularly when considering a 2 dimensional Markov chain, using the successive lumping approach, cf. [12]. In order to find the steady state distribution one can use this approach to compute the rate matrix R . In many cases calculating this matrix using successive lumping requires the inversion of a matrix with the structure of B . Examples can be found in queueing (an $E_k/M/c$ queue with batch service), reliability systems and inventory models (a inventory model with batch arrivals and random lead time). Specific about these systems an the complete procedure to compute the steady state distribution can be found in [11]. Below we will discuss other (classes of) applications in which this type of matrix arises naturally or can be constructed.

4.1 A general non-transient Markov chain

Let Q be the transition rate matrix of a general - non-transient Markov chain and let B be the matrix with elements

$$b(i, j) = \begin{cases} q(0, 0) - 1 & \text{if } (i, j) = (0, 0), \\ q(i, j) & \text{otherwise,} \end{cases}$$

i.e., $B = Q - \delta' \cdot \delta$. The solution to the steady state equations:

$$\begin{aligned} \pi Q &= 0 \\ \pi \mathbf{1} &= 1 \end{aligned}$$

is given by: $\pi = \frac{\delta B^{-1}}{\delta B^{-1} \mathbf{1}}$. Algorithm 1, can be used to obtain the solution efficiently and in case of a homogenous process Algorithm 2 should be applied.

4.2 A birth-and-death process with an absorbing state

In this section, we consider a special case of the structure described above that has applications to a birth-and-death process with abandonments. These models occur for example in diffusion processes (cf. [10]) and in randomly changing environments (cf. [6]). We assume that in every state there is an (equal) positive rate to leave the system and go to an absorbing state (state 0). The remaining states form a birth-and-death process. This process is element homogenous according to its description in Definition 1, but with the difference that $b_0^u = 0$ and $b_1^d = 0$. Therefore the transition matrix has the following form:

$$B = \begin{bmatrix} -b^d & 0 & 0 & 0 & 0 & \dots \\ b^z & -b^z - b^u & b^u & 0 & 0 & \dots \\ b^z & b^d & -b^w & b^u & 0 & \ddots \\ b^z & 0 & b^d & -b^w & b^u & \ddots \\ b^z & 0 & 0 & b^d & -b^w & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad (36)$$

where options: 1: $b_i^z = b^z$ for all i or the original B with $b_0^u = 0$

The procedure to find the inverse of B is described in Algorithm 3 below. This is a special case of Algorithm 2, that requires element homogeneity; only the first row of this matrix slightly does not meet this requirement.

Algorithm 3.

At stage 1:

- a) Calculate γ using Eq. (22).
- b) Calculate ψ using Eq. (26).
- c) The 0^{th} column of C is computed using Eq. (8) of Proposition 1.
- d) The remaining elements of the 0^{th} row of C are 0, as is described in Proposition 16.
- e) Calculate element $c(1, 1)$ by Proposition 17.
- f) The remaining elements ($j \geq 2$) of the first row are: $c(1, j) = \gamma^{j-1}c(1, 1)$.

At stage $i = 1, 2, \dots, \ell + 1$:

- a) The diagonal element $c(i, i)$ is computed by Eq. (27).
- b) The elements of C_i , the i^{th} row of C to the right of the diagonal are computed using Proposition 7 and in particular Eq. (23) in that proposition.
- c) The under diagonal elements of C'_i , the i^{th} column of C are computed using Eq. (25) of Proposition 8.

Most of the steps above are similar to those in Algorithm 2, and the two Propositions below justify the remaining steps.

Proposition 16. *The elements $c(0, j)$ are equal to zero for all $j \geq 1$.*

Proof. This result follows immediately from the fact that all elements are nonpositive and by considering $B_1 C'_0 = b^d(-1/b^d) + z \sum_{j=1}^{\infty} c(0, j) = 1$. \square

Proposition 17. *The element $c(1, 1)$ can be calculated as follows:*

$$c(1, 1) = \frac{1}{-b^z - b^u + b^d \gamma}.$$

Proof. We consider the equation $C_1 B'_1 = 1$. This gives us:

$$(-b^z - b^u)c(1, 1) + b^d c(1, 2) = 1.$$

Because $c(1, 2) = \gamma c(1, 1)$ we derive:

$$c(1, 1) = \frac{1}{(-b^z - b^u) + b^d \gamma}.$$

\square

4.3 Value Functions

In many models, e.g., to compute the value function of the expected α -discounted cost associated with a continuous time Markov chain, one needs to solve equations of the following type:

$$\alpha V = c + QV \quad (37)$$

where c is the cost rate function defined on the state space and V the unknown value function to be determined as a solution of Eq. (37). We refer to [19], [18] for more background on these models. The solution to this equation is

$$V = -Q_\alpha^{-1}c \quad (38)$$

where $Q_\alpha = (Q - \alpha I)$ is a rate matrix of a transient Markov Process where in addition to the rates specified by Q one has introduced an additional event of ‘exiting’ or ‘halting’ the process at a rate α .

When Q has a tridiagonal form, i.e. is a birth-and-death process, we can use the algorithm of the previous section to compute this inverse of matrix Q_α .

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